Lecture 7: Change of Parameters: Differentiable Functions on Surfaces

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Math 142:
Differential Geometry

Introduction

Big Ideas

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- According to our definition, each point p of a regular surface belongs to a coordinate neighborhood.
- The points of such a neighborhood are characterized by their coordinates, and we should be able, therefore, to define the local properties which interest us in terms of these coordinates.
- For example, it is important that we be able to define what it means for a function $f : S \to \mathbb{R}$ to be differentiable at a point p of a regular surface S.

Differentiability

Potential Problems with Parametrizations

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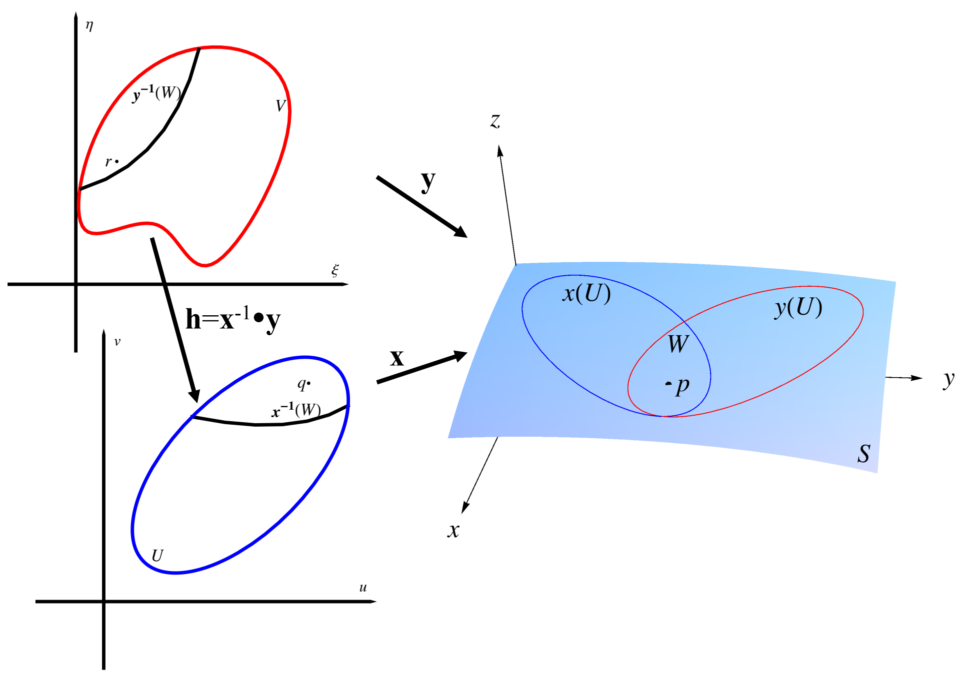
- However, the same point of S can belong to various coordinate neighborhoods (in the sphere example, any point of the interior of the first octant belongs to three of the six given coordinate systems).
- For the above definition to make sense, it is necessary that it does not depend on the chosen system of coordinates. In other words, it must be shown that when p belongs to two coordinate neighborhoods, with parameters (u, v) and (ξ, η), it is possible to pass from one of these pairs of coordinates to the other by means of a differentiable transformation.

Change of Parameters

Proposition (∗)

Let p be a point of a regular surface S, and let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$, $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the "change of coordinates" $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.



Differentiable Functions on a Surface

Definition

Let $f : V \subset S \to \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be *differentiable at* $p \in V$ if, for some parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\mathbf{x}^{-1}(p)$. f is *differentiable in* V if it is differentiable at all points of V.

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Remark

We shall frequently make the notational abuse of indicating f and $f \circ \mathbf{x}$ by the same symbol $f(u, v)$, and say that $f(u, v)$ is the *expression* of f in the system of coordinates $\mathbf{x}$. This is equivalent to identifying $\mathbf{x}(U)$ with U and thinking of (u, v), indifferently, as a point of U and as a point of $\mathbf{x}(U)$ with coordinates (u, v). From now on, abuses of language of this type will be used without further comment.

Examples

Example

Let S be a regular surface and $V \subset \mathbb{R}^3$ be an open set such that $S \subset V$. Let $f : V \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then the restriction of f to S is a differentiable function on S. In fact, for any $p \in S$ and any parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ in p, the function $f \circ \mathbf{x} : U \to \mathbb{R}$ is differentiable.

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1. The *height function* relative to a unit vector $v \in \mathbb{R}^3$, $h : S \to \mathbb{R}$, given by $h(p) = p \cdot v$, $p \in S$, where the dot denotes the usual inner product in $\mathbb{R}^3$. $h(p)$ is the heigh of $p \in S$ relative to a plane normal to v and passing through the origin of $\mathbb{R}^3$.

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2. The square of the distance from a fixed point $p_0 \in \mathbb{R}^3$, $f(p) = |p - p_0|^2$, $p \in S$. The need for taking the square comes from the fact that the distance $|p - p_0|$ is not differentiable at $p = p_0$.

Differentiable Functions Between Surfaces

The definition of differentiability can be easily extended to mappings between surfaces. A continuous map $\varphi : V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is said to be *differentiable at* $p \in V$ if, given parametrizations

$$\mathbf{x}_1 : U_1 \subset \mathbb{R}^2 \to S_1, \quad \mathbf{x}_2 : U_2 \subset \mathbb{R}^2 \to S_2,$$

with $p \in \mathbf{x}_1(U)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

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- In other words, φ is differentiable if when expressed in local coordinates as $\varphi(u_1, v_1) = (\varphi_1(u_1, v_1), \varphi_2(u_1, v_1))$, the functions φ_1 and φ_2 have continuous partial derivatives of all orders.

Facts and Remarks

Remark

The proof of Proposition $(*)$ makes essential use of the fact that the inverse of a parametrization is continuous. Since we need $(*)$ to be able to define differentiable functions on surfaces (a vital concept), we cannot dispose of this condition in the definition of a regular surface.

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U and $\mathbf{x}(U)$ are diffeomorphic (i.e., every regular surface is locally diffeomorphic to a plane) and justifies the identification made above. This is equivalent to:

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Remark

Proposition $(*)$ implies that a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ is a diffeomorphism of U onto $\mathbf{x}(U)$. Actually, we can now characterize the regular surfaces as those subsets $S \subset \mathbb{R}^3$ which are locally diffeomorphic to $\mathbb{R}^2$; that is, for each point $p \in S$, there exists a neighborhood V of p in S, an open set $U \subset \mathbb{R}^2$, and a map $\mathbf{x} : U \to V$, which is a diffeomorphism.

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Let S_1 and S_2 be regular surfaces. Assume that $S_1 \subset V \subset \mathbb{R}^3$, where V is an open set of $\mathbb{R}^3$, and that $\varphi : V \to \mathbb{R}^3$ is a differentiable map such that $\varphi(S_1) \subset S_2$. Then the restriction $\varphi|S_1 : S_1 \to S_2$ is a differentiable map.

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1. Let S be symmetric relative to the xy plane; that is, if $(x, y, z) \in S$, then also $(x, y, -z) \in S$. Then the map $\sigma : S \to S$, which takes $p \in S$ into its symmetrical point, is differentiable, since it is the restriction to S of $\sigma : \mathbb{R}^3 \to \mathbb{R}^3$, $\sigma(x, y, z) = (x, y, -z)$. This, of course, generalizes to surfaces symmetric relative to any plane of $\mathbb{R}^3$.

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2. Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation of angle θ about the z axis, and let $S \subset \mathbb{R}^3$ be a regular surface invariant by this rotation; i.e., if $p \in S$, $R_{z,\theta}(p) \in S$. Then the restriction $R_{z,\theta} : S \to S$ is a differentiable map.

Diffeomorphisms

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- A mapping $\varphi : U \subset S_1 \to S_2$ is a *local diffeomorphism* at $p \in U$ if there exists a neighborhood $V \subset U$ of p such that φ restricted to V is a diffeomorphism onto an open set $\varphi(V) \subset S_2$.

Examples

Example

Show that the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and the ellipsoid

$$E = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

are diffeomorphic.